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# Entanglement properties and momentum distributions of hard-core anyons on a ring.

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We study the one-particle von Neumann entropy of a system of $N$ hard-core anyons on a ring. The entropy is found to have a clear dependence on the anyonic parameter which characterizes the generalized fractional statistics described by the anyons. This confirms the entanglement is a valuable quantity to investigate topological properties of quantum states. We derive the generalization to anyonic statistics of the Lenard formula for the one-particle density matrix of $N$ hard-core bosons in the large $N$ limit and extend our results by a numerical analysis of the entanglement entropy, providing additional insight into the problem under consideration.

In recent years an intense research activity has been devoted to the study of entanglement in many-body states. Initially, this effort has been mostly motivated by the fact that quantum correlated many-body states, which appear in various solid-state models, can be valuable resources for information processing and quantum computation [1, 2]. The theory of entanglement is now attracting even more attention because of its fundamental implication for the development of new efficient numerical methods for quantum systems [3, 4, 5] and for the characterization of quantum critical phases [6, 7, 8].

Generally speaking, entanglement measures nonlocal properties of composite quantum systems and it can provide additional information to that obtained by investigating local observables or traditional correlation functions. In this respect entanglement might be a sensitive probe into the topological properties of quantum states. A particularly significant quantity is the entanglement entropy $S_A$, which is defined in a bipartite system $A \cup B$ and quantified as the von Neumann entropy $S_A = -\text{Tr}\rho_A \ln \rho_A$ associated to the reduced density matrix $\rho_A$ of a subsystem $A$. In two-dimensional systems a firm connection between topological order and entanglement entropy has been established in [9, 10], where the entanglement entropy was defined by spatial partitioning. Recent studies on Laughlin states [11, 12] have considered the entanglement entropy associated to particle partitioning [11, 12]. Also in this case, the entanglement entropy turns out to reveal important aspects of the topological order in Laughlin States.

The two-dimensional case is of particular interest due to the existence of models whose elementary excitations exhibit generalized fractional statistics. Anyons, the particles obeying such statistics, play a fundamental role in the description of the fractional quantum Hall effect [13]. Although this concept is essentially two-dimensional, anyons can also occur in one-dimensional (1D) systems [14, 15, 16, 17, 18, 19, 20], where statistics and interactions are inextricable, leading to strong short-range correlations. The 1D anyonic models have proven useful to study persistent charge and magnetic currents in 1D mesoscopic rings [15]. This possibility and their own pure theoretical interest lead us to investigate the effects of the anyonic statistics on the entanglement entropy in the present Letter. A discussion about quantum statistics and entanglement in a two-fermions system was introduced in [21] and extended to the case of two bosons in [22]. A mechanism of spin-space entanglement transfer based on the indistinguishability of two particle was proposed in [23] and shown to depend on the statistics (either fermionic or bosonic) of the particles involved. A comparison between the entanglement of two bosons and two fermions confined in a 1D harmonic trap has been presented recently in [24].

In this Letter, we consider a system of $N$ hard-core anyons on a ring which is the direct anyonic generalization of the Tonks-Girardeau gas. It offers a convenient framework to study topological effects: the many body ground state is known and its behaviour under the exchange of two particles interpolates between bosons and fermions. We carry out an analytical and numerical analysis of the dependence of the one-particle von Neumann entropy on the statistical parameter which determines the symmetry of the many body state. We derive the large $N$ asymptotic expression of the anyonic one-particle density matrix. This asymptotic form generalizes the one obtained for hard-core bosons [25, 26] and provides a one-parameter family of zero temperature momentum distributions interpolating between hard-core boson and free fermion distributions. Our results show that particle entanglement depends in a non-trivial manner on the statistics and as such, may prove to be relevant to the study of topological properties of many-body quantum states.

Let us consider a 1D system of anyons confined on a ring of length $L$ interacting each other via a repul-

sive $\delta-$function potential. The model is defined by the Hamiltonian:

$$H = -\sum_i^N \frac{\partial^2}{\partial x_i^2} + \gamma \sum_{1\le i<j\le N} \delta(x_i - x_j), \qquad (1)$$

The $N-$anyons wave function $\Psi^\theta(x_1, x_2, .., x_N)$ exhibits a generalized symmetry under the exchange of particles:

$$\Psi^\theta(\cdots x_i, x_{i+1}\cdots) = -e^{i\theta\varepsilon(x_{i+1}-x_i)}\Psi^\theta(\cdots x_{i+1}, x_i\cdots), \qquad (2)$$

where $\varepsilon(x) = 0$ (or 1) if $x > 0$ ($x < 0$) and $\theta$ is the anyonic parameter, defined as in Ref. [15]. For $\theta = 0$ this model describes free fermions while, for $\theta = \pi$, it reduces to the Lieb-Liniger Bose gas.

As first discussed in [16], the problem of 1D anyons with contact interactions allows for an exact Bethe ansatz solution which shows that the Hamiltonian (1) has the same energy spectrum than a 1D interacting Bose gas with anyonic statistics-dependent effective coupling in the moving frame. Very recently, a detailed analysis of the low-energy properties of this model has been carried out in [17, 18, 19]. It was shown that the low-temperature thermodynamics of 1D anyons with a $\delta$-function potential coincides with the one of a gas of ideal particles obeying Haldane statistics: the interplay between the anyonic parameter $\theta$ and the coupling constant $\gamma$ determines a continuous range of these generalized statistics. These studies have shown that, for strong coupling, the dispersion relations of the anyon gas remain linear in the thermodynamic limit and the finite size corrections of the ground state energy found a central charge $c = 1$.

In the case of spatial partitioning, the subsystem $A$ being a block of size $l$, conformal field theory results [27, 28] predict the entanglement entropy (block entropy) $S_A(l)$ to scale as $S_A(l) \sim \frac{1}{3}\ln l$. The dependence on the coupling constant and on the anyonic parameter is expected to show up in the sub-leading terms which, to the best of our knowledge, remain unknown. Below we will demonstrate that the one-particle entanglement entropy $S_1^\theta(N)$ of $N-$anyons depends clearly on the anyonic parameter. Furthermore, we will show that, for a great number N of particles, the dependence of the entanglement entropy on the anyonic statistics appears in the sub-leading term.

Let us consider the limit of hard-core anyons, *i.e.* $\gamma \to \infty$. As recently shown in [20], the Fermi-Bose mapping method for one dimensional hard-core bosons [32] can be generalized to an anyon-fermion mapping (AF). Imposing the exclusion principle, *i.e.* the vanishing of the many-body wave function when two particles occupy the same position, the AF mapping reads [20]:

$$\Psi_0^\theta(x_1,\cdots,x_N) = \left[\prod_{1\le i<j\le N} A(x_i,x_j)\right]\Psi_0^F(x_1,\cdots,x_N), \qquad (3)$$

where $\Psi_0^F(x_1,\cdots,x_N)$ is the $N$ free fermion ground state function and $A(x_i,x_j) = e^{i\theta\varepsilon(x_i,x_j)}$. In the following, we restrict ourselves to the case where $N$ is odd, which corresponds to a non-degenerate ground state. The topological properties of the $N$-anyon wave function are encoded in the factor $\prod_{1\le i<j\le N} A(x_i,x_j)$ which gives the statistical phase $e^{i\theta P}$ resulting from the $P$ exchanges needed for the particle positions to be brought to the ordering $0 \le x_1 \le x_2 \le \cdots \le x_N \le L$. The periodic boundary conditions $\Psi^\theta(x_1,\cdots,x_i+L,\cdots,x_N) = \Psi^\theta(x_1,\cdots,x_i,\cdots,x_N)$ impose the anyonic parameter to be an integer multiple of $2\pi/(N-1)$, $\theta = \frac{2\pi}{N-1}n$.

We are interested in computing the entropy $S_1^\theta(N) = -\mathrm{Tr}(\rho_N^\theta \ln \rho_N^\theta)$, where $\rho_N^\theta(x-x')$ is the one-particle reduced density matrix:

$$\rho_N^\theta(x-x') = \int_0^L \cdots \int_0^L \prod_{i=2}^N dx_i \left[\bar{\Psi}^\theta(x,x_2,\cdots,x_N) \times \Psi^\theta(x',x_2,\cdots,x_N)\right], \qquad (4)$$

normalized such that $\rho_N(0) = 1$. In momentum space, the entanglement entropy simply reads:

$$S_1^\theta(N) = -\sum_{n=-\infty}^{\infty} c_N^\theta(n)\ln c_N^\theta(n), \qquad (5)$$

where $c_N^\theta(n) = 1/L\int_0^L \rho_N^\theta(x)\cos(2\pi/Lnx)$ is the momentum occupation in the ground state. For $\theta = 0$ (free fermions), one has $c_N^0(n) = 1/N$ for $(N-1)/2 \le n \le (N+1)/2$ leading to the well known-result for free fermions $S_1^0 = \ln N$. The von Neumann entropy actually measures the uncertainty about the quantum state to attribute a state to the subsystem in consideration, which in this case comes exclusively from the fact the fermions are indistinguishable.

The mapping (3) between Fermi and anyon eigenfunctions preserves all scalar products and thus the energy spectrum and all the probability distributions involving the norm of the wave function are identical. Nevertheless, $c_N^\theta(n)$ strongly depends on $\theta$ as can be guessed from the drastic difference between the momentum distributions of free fermions and hard-core bosons.

Analytical expressions of $c^\pi(n)$ for small ($|n| \ll 1$) and large ($|n| \gg 1$) momenta have been found in the thermodynamic limit [33, 34, 35, 36, 37]. These results show a $|n|^{-1/2}$ singularity at $n = 0$, reflecting the tendency towards Bose-Einstein condensation. The corresponding result for finite $N$ is more cumbersome. Using the $N \gg 1$ asymptotic result for $\rho^\pi(x)$ [25],

$$\rho^\pi(x) \sim \rho_\infty N^{-\frac{1}{2}}|\sin\pi x/L|^{-\frac{1}{2}}, \qquad (6)$$

with $\rho_\infty = G(3/2)^4/\sqrt{2}$ and $G(z)$ the Barnes G-function [40], $c_n(N)$ for $N \gg n$ was shown to behave like [26]:

$$c_n^\pi(N) \sim \frac{\rho_\infty}{\sqrt{\pi}}\frac{\Gamma(n+1/4)}{\Gamma(n+3/4)}N^{-\frac{1}{2}}, \qquad (7)$$

where $\Gamma(z)$ is the standard Gamma function. We generalized the above results to anyonic statistics. Representing the density matrix (4) in terms of a Toeplitz $N-1 \times N-1$ determinant:

$$N\rho_N^\theta(x) = \det_{N-1}\left[\phi_{k,l}\right](x) \tag{8}$$

where

$$\phi_{k,l} = \int_0^{2\pi} ds \frac{2e^{i(k-l)s}}{\pi} A(s-\frac{2\pi x}{L}) \sin\left(\frac{s}{2} - \frac{\pi x}{L}\right) \sin\left(\frac{s}{2}\right), \tag{9}$$

we were able to compute the asymptotic form of $\rho_N^\theta$ using the Fisher-Hartwig conjecture [38, 39]. For $N \gg 1$ the one-particle anyon density matrix reads:

$$\begin{aligned}\rho_N^\theta(x) \sim \quad & (2N)^{\alpha(\theta)-1} G\left(1+\frac{\theta}{2\pi}\right)^2 G\left(2-\frac{\theta}{2\pi}\right)^2 \\ & \times e^{i(\frac{\theta}{2\pi}-\frac{1}{2})} e^{-iN(\frac{\theta}{\pi}-1)\pi x/L-} \left|\sin\left(\frac{\pi x}{L}\right)\right|^{\alpha(\theta)-1}\end{aligned} \tag{10}$$

and $\alpha(\theta) = \frac{\theta}{\pi}(1-\frac{\theta}{2\pi})$. We see that Eq. (6) is recovered for $\theta = \pi$. Details of the derivation and a more complete discussion of this result will be presented elsewhere [41, 42]. In the case of the generating function (9) the analysis of the behaviour of the Toeplitz determinant is subtle and the Fisher-Hartwig formula remains a conjecture. The validity of Eq. (10) has thus be compared to the numerical evaluation of the determinant (8) for finite $N$. For small $\theta$, the convergence to the asymptotic result is quite slow and the formula provides a rough estimate for finite $N$, as can be seen for N=121 and $\theta = \pi/60$ (Fig. 1 (a)). The likeness increases greatly with $\theta$ and the agreement is already perfect with $\theta = \pi/2$ for $N = 61$ (Fig. 1 (b)) and for $\theta = 9\pi/10$, close to hard-core bosons, for N=21 (Fig. 1 (c)).

The Fourier coefficients of (10) can be computed analytically [44]. The asymptotic behaviour of $c_n^\theta(N)$ for $N \gg n$ reads:

$$\begin{aligned}c_n^\theta(N) \sim \quad & \frac{1}{\pi} N^{\alpha(\theta)-1} G\left(1+\frac{\theta}{2\pi}\right)^2 G\left(2-\frac{\theta}{2\pi}\right)^2 \\ & \times\Gamma\left(\alpha(\theta)\right)\sin\left(\pi(1-\alpha(\theta)\right) \\ & \times\frac{\Gamma(n'+\theta^2/(4\pi^2))}{\Gamma(n'+\theta/\pi-\theta^2/(4\pi^2))}.\end{aligned} \tag{11}$$

where $n' = n - \lfloor N(\theta/(2\pi) - 1/2) + \theta/(4\pi) + 3/4 \rfloor$ with $\lfloor x \rfloor$ being the integer part of $x$. A comparison between the above formula and the exact results is shown in Fig. 2. Note that, from Eq.(10), the number of particles occupying the low-energy state ($n' = 0$) scales as $N^{\alpha(\theta)-1}$, thus ruling out the possibility of anyon-condensation predicted in the case of free anyons [45]. For $n \gg N$, the bosonic momentum distribution $c_n^\pi$ decays like $n^{-4}$ [26, 46, 47]. Based on exact results for $N = 5, 7, 9$ and on numerical computations (see Fig. 2), we expect this to be true for anyonic statistics as well. Under this assumption, the terms $c_n^\theta(N)$ for $n \gg N$ will not contribute

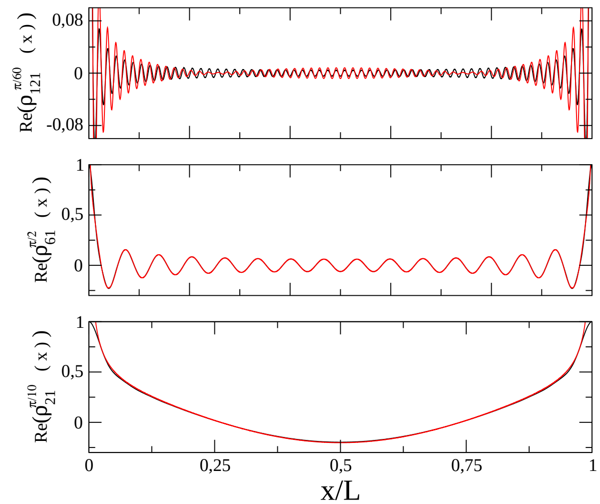


FIG. 1: (Color online) Comparison between the numerical computation (red/dark gray curve) and asymptotic equation (black curves) for the one-particle anyon density for: (a): $N = 121, \theta = \pi/60$, (b): $N = 61, \theta = \pi/2$ and (c): $N = 21, \theta = 9\pi/10$.

significantly to the entanglement entropy. While the main contribution to $S_1^\theta(N)$ can be extracted from (11), the numerical simulations show that, for finite $N$, the crossover region between the asymptotic (11) and power-law can not be neglected, especially near the fermionic point $\theta = 0$.

We have determined numerically the value of the one-particle von Neumann entropy $S_1^\theta(N)$ for different values of $N$ (Fig. 3 (a)-(b)) which can be shown to scale with $N$ in the following way:

$$S_1^\theta(N) \approx \ln N + f(\theta) + \frac{\kappa(\theta)}{\sqrt{N}}. \tag{12}$$

The anyonic-parameter-dependent function $f(\theta)$ was determined numerically for system sizes for which the last term of this expansion can not be neglected. However, in the thermodynamic limit, only $f(\theta)$ will remain relevant in our discussion. The result of our numerical analysis are displayed in Fig. 3, where panel (a) is the data as fitted by Eq. (12) for three values of the anyonic param eter and panel (b) is the resulting $f(\theta)$. This function decreases monotonically from free fermions to hard-core bosons where it respectively takes the value $f(0) = 0$ and $s(\pi) \approx -0.3$, and from Fig. 3-(b) it can be seen to be well fitted by a sinus function.

To conclude, we have investigated analytically and numerically the one-particle von Neumann entropy and the momentum distributions of $N$ hard-core anyons on a ring. We have determined the asymptotic expressions for the one-particle density matrix and for the momentum distributions. Numerical results show the entanglement entropy exhibits a simple and non-trivial dependence on the

anyonic parameter even for $N \to \infty$, making it a suitable tool to study the topological properties of many-body quantum states.

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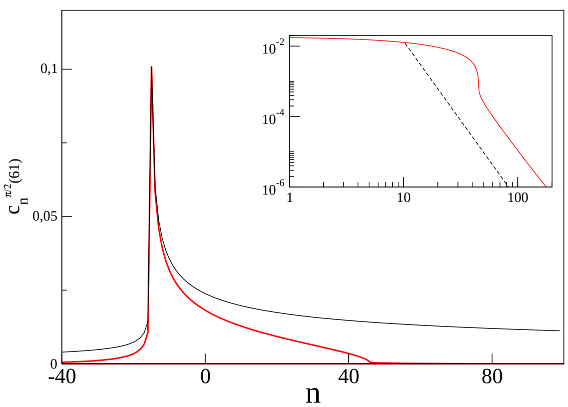


FIG. 2: $c_n^\theta(N)$ obtained from numerical (Color online) and Eq.(11) (Black line) versus $n$ for $\theta = \pi/2$ and $N = 61$. The inset is the same plot for $n > 0$ in a log-log scale (red/dark gray curve), while the dashed curve is a visual guide proportional to $n^{-4}$.

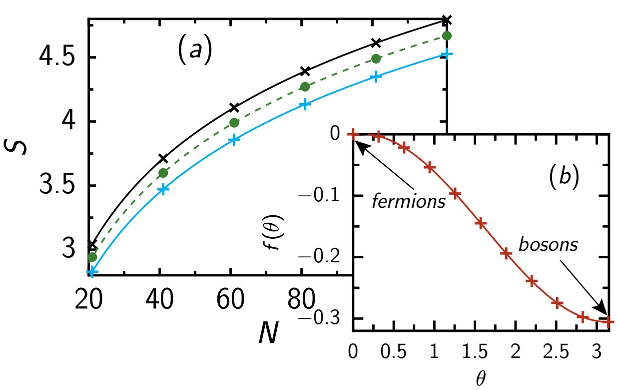


FIG. 3: (Color online) (a): Entanglement entropy as a function of N for $\theta = \pi$ (pluses), $\theta = \pi/2$ (crosses), $\theta = \pi/10$ (dots) in log-linear scale, fitted according to Eq. (12). (b): $f(\theta)$ obtained by numerical integration (pluses) and the corresponding sinus fit (plain line).

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